(Defn.) Weak Convergence:- Let X be a normed space, X^* and X^{**} be the first and second dual spaces of X respectively.

(i) A sequence $\{x_n\}$ in X is called weakly convergent in X, in symbols $x_n \xrightarrow{W} x$, if there exists an element $x \in X$ such that $\underset{n\to\infty}{Lt.} |f(x_n) - f(x)| = 0 \ \forall f \in X^*$, i.e., for $\epsilon > 0$ there exists a natural number N such that $|f(x_n) - f(x)| \leq \epsilon$ for $n > N$ and $\forall f \in X^*$.

(ii) A sequence $\{f_n\}$ in X^* is called weakly * convergent to f in X^* if $\underset{n\to\infty}{Lt.} |f_n(x) - f(x)| = 0 \ \forall x \in X$.

M.U. M.Sc. 95 Q.No.- Show that the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent if $x = \{x_1, x_2, \ldots, x_n \ldots\}$ and $y = \{y_1, y_2, \ldots y_n \ldots\}$ are elements of the space l_2.

Soln.- Since $x, y \in l_2$, the series $\sum |x_n|^2$ & $\sum |y_n|^2$ are convergent —— (1)

For each +ve integer

$$(|x_n| - |y_n|)^2 \geq 0.$$

So that $2|x_n|\,|y_n| \leq |x_n|^2 + |y_n|^2$ —— (2)

Now, $\sum(|x_n|^2 + |y_n|^2)$ is convergent by (1). Hence by comparison test, it follows from (2), that,

$\sum 2|x_n|\,|y_n|$ is convergent,

and therefore, $\sum |x_n|\,|y_n|$ is convergent —— (3)

Since, $|x_n y_n| = |x_n|\,|y_n|$

Therefore, $\sum |x_n y_n|$ is Convergent from
(3).

This implies that $\sum x_n y_n$ is absolutely Convergent, and hence Convergent.

Ex.- If a Sequence (N_K) of normal operators on H Converges to an operator N on H then N is normal.

<u>Verification</u>:- $\because N_K \Rightarrow N$, $\|N_K - N\| \to 0$ as $K \to \infty$

Therefore, $\|N_K^* - N^*\| = \|(N_K - N)^*\| = \|N_K - N\| \to 0$

Hence, $N_K^* \to N^*$.

$$\text{Now, } \|NN^* - N^*N\| = \|NN^* - N_K N_K^* + N_K N_K^* - N_K^* N_K + N_K^* N_K - N^* N\|$$
$$\leq \|NN^* - N_K N_K^*\| + \|N_K N_K^* - N_K^* N_K\| + \|N_K^* N_K - N^* N\|$$
$$= \|NN^* - N_K N_K^*\| + \|N_K^* N_K - N^* N\|$$
$$[\because N_K N_K^* = N_K^* N_K] \to 0$$

$\therefore NN^* = N^*N$

Hence N is a normal operator.

=0=

Dr. Sunil Jee

S.B. College, Ara.